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# PRINCIPLE OF CORRESPONDENCE OF STATIC BOUNDARY-VALUE PROBLEMS OF NONLINEAR VISCOELASTICITY WITH AGING TO BOUNDARY-VALUE PROBLEMS OF THE THEORY OF ELASTICITY

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*The principle of correspondence of boundary-value problems of the nonlinear nonuniform anisotropic theory of viscoelasticity to boundary-value problems of the theory of elasticity is formulated. The correspondence is established by means of integral transforms with previously unknown kernels. A class of viscoelastic materials for which these transforms can be reduced to boundary-value problems of fictitious elasticity is determined.*

We consider a nonlinear static boundary-value problem of nonuniform anisotropic viscoelasticity with aging [1]:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0, \quad \varepsilon_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\},$$

$$\sigma_{ij}(x,t) = \sum_{n=1}^{\infty} \underbrace{\int_0^t \ldots \int_0^t}_{n} R_{(ij)}(x,t,\tau_1,\tau_2,\ldots,\tau_n) \prod_{m=1}^{n} \varepsilon_{i_m j_m}(\tau_m)\, d\tau_m, \tag{1}$$

$$\sigma_{ij}(x,t)n_j = P_i(x,t), \quad x \in S_\sigma, \quad u_i(x,t) = u_i^0(x,t), \quad x \in S_u.$$

Here $\varepsilon_{ij}(x,t)$ and $\sigma_{ij}(x,t)$ are the components of the strain and stress tensors, $R_{(ij)}(x,t,\tau_1,\ldots,\tau_n)$ are the relaxation kernels, $(ij) = i_1j_1i_2j_2\ldots i_nj_n$, $u_i(x,t)$ are the components of the displacement vector, $P_i(x,t)$ are the components of the stress vector on a part of the surface of the body $S_\sigma$ with the unit normal $n$ ($n_i$ are its components), and $u_i^0(x,t)$ are the components of the displacement vector on a part of the surface $S_u$.

We introduce linear integral transforms with the kernel $\varphi_l(p,t)$:

$$f(t) = \int_{\Omega_p} f^*(p)\varphi_l(p,t)\, dp. \tag{2}$$

Here the functions $f(t)$ and $f^*(p)$ are the integral image and the inverse integral image, respectively. We assume that the following inverse integral transforms are known for the integral transforms (2)

$$f^*(p) = \int_{\Omega_t} f(t)\varphi_l^+(p,t)\, dt, \tag{3}$$

where $\varphi_l^+(p,t)$ are the resolvent kernels.

Thus, for each inverse image $f^*(p)$, Eqs. (2) and (3) make it possible to choose as many possible integral images $f(t)$ as desired by choosing a specific form of the kernel $\varphi_l(p,t)$. The same two equations also allow us to find as many integral inverse transforms as desired for each image $f(t)$. Here the subscript $l$ in the notation used for the kernels of the direct and inverse integral transforms make it possible to distinguish one integral transform from another. Using Eqs. (2) and (3), we can obtain

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$$f(t) = \int_{\Omega_p} \left[ \int_{\Omega_{t'}} f(t')\varphi_l^+(p,t')\,dt' \right] \varphi_l(p,t)\,dp. \tag{4}$$

By changing the order of integration in (4), we have

$$f(t) = \int_{\Omega_{t'}} f(t') \left[ \int_{\Omega_p} \varphi_l^+(p,t')\varphi_l(p,t)\,dp \right] dt'. \tag{5}$$

In the last formula, $\int_{\Omega_p} \varphi_l^+(p,t')\varphi_l(p,t)\,dp = G(t',t)$ is, by virtue of (5), the kernel of the following unitary integral transform

$$f(t) = \int_{\Omega_{t'}} G(t',t)f(t')\,dt'.$$

We further assume that all of the integral transforms are one-to-one and continuous and that it is possible to change the order of integration and to calculate multiple integrals in an arbitrary order.

The foregoing permits us to prove the following theorem.

**Theorem 1.** *Boundary-value problem* (1) *in inverse integral images* (2) *has the form*

$$\frac{\partial \sigma_{ij}^*(x,p)}{\partial x_j} + X_i^*(x,p) = 0, \quad \varepsilon_{ij}^*(x,p) = \frac{1}{2}\left\{ \frac{\partial u_i^*(x,p)}{\partial x_j} + \frac{\partial u_j^*(x,p)}{\partial x_i} \right\},$$

$$\sigma_{ij}^*(x,p) = \sum_{n=1}^{\infty} \underbrace{\int_{\Omega_{p_1}} \cdots \int_{\Omega_{p_n}}}_{n} R_{(ij)}^*(x,p,p_1,\ldots,p_n) \prod_{\xi=1}^{n} \varepsilon_{i_\xi j_\xi}^*(x,p_\xi)\,dp_\xi,$$

$$\sigma_{ij}^*(x,p)n_j(x) = P_i^*(x,p), \quad x \in S_\sigma, \qquad u_i^*(x,p) = u_i^0(x,p), \quad x \in S_u.$$

**Proof.** Let the components of the displacement vector $u_i(x,t)$ be the integral images of the inverse images $u_i^*(x,p)$:

$$u_i(x,t) = \int_{\Omega_p} u_i^*(x,p)\varphi_l(p,t)\,dp. \tag{6}$$

Then the Cauchy relations make it possible to obtain

$$\varepsilon_{ij}(x,t) = \frac{1}{2}\left\{ \frac{\partial}{\partial x_j} \int_{\Omega_p} u_i^*(x,p)\varphi_l(p,t)\,dp + \frac{\partial}{\partial x_j} \int_{\Omega_p} u_j^*(x,p)\varphi_l(p,t)\,dp \right\}. \tag{7}$$

In (7), differentiation and integration are performed with different parameters. Thus, by changing the order in which these operations are performed, we have

$$\varepsilon_{ij}(x,t) = \int_{\Omega_p} \left\{ \frac{1}{2}\left[ \frac{\partial u_i^*(x,p)}{\partial x_j} + \frac{\partial u_j^*(x,p)}{\partial x_i} \right] \right\} \varphi_l(p,t)\,dp. \tag{8}$$

Let $\varepsilon_{ij}(x,t)$ in relations (1) be the integral image of the quantity $\varepsilon_{ij}^*(x,p)$:

$$\varepsilon_{ij}(x,t) = \int_{\Omega_p} \varepsilon_{ij}^*(x,p)\varphi_l(p,t)\,dp. \tag{9}$$

Then we can use (8) and (9) to obtain

$$\varepsilon_{ij}^*(x,p) = \frac{1}{2}\left\{ \frac{\partial u_i^*(x,p)}{\partial x_j} + \frac{\partial u_j^*(x,p)}{\partial x_i} \right\}. \tag{10}$$

Also taking the relaxation kernels $R_{(ij)}(x,t,\tau_1,\ldots,\tau_n)$ to be the integral images of the quantities $R_{(ij)}^*(x,t,\tau_1,\ldots,\tau_{n-1},p_1)$, we have

$$R_{(ij)}(x,t,\tau_1,\ldots,\tau_n) = \int_{\Omega_{p_1}} R_{(ij)}^*(x,t,\tau_1,\ldots,\tau_{n-1},p_1)\varphi_l^+(p_1,\tau_n)\,dp_1. \tag{11}$$

In turn, the function $R^{*}_{(ij)}(x,t,\tau_1,\ldots,\tau_{n-1},p_1)$ in Eq. (11) can be considered the integral image of the function $R^{**}_{(ij)}(x,t,\tau_1,\ldots,\tau_{n-2},p_2,p_1)$:

$$R^{*}_{(ij)}(x,t,\tau_1,\ldots,\tau_{n-1},p_1)=\int\limits_{\Omega_{p_2}} R^{**}_{(ij)}(x,t,\tau_1,\ldots,\tau_{n-2},p_2,p_1)\varphi_l^{+}(p_2,\tau_{n-1})\,dp_2.$$

Thus, with allowance for the above, we represent the function $R_{(ij)}(x,t,\tau_1,\ldots,\tau_n)$ in the form

$$R_{(ij)}(x,t,\tau_1,\ldots,\tau_n)=\underbrace{\int\limits_{\Omega_p}\cdots\int\limits_{\Omega_{pn}}}_{n+1} R^{(*)}_{(ij)}(x,p,p_1,\ldots,p_n)\varphi_{l_1}(p,t)\prod_{m=1}^{n}\varphi_l^{+}(p_m,\tau_m)\,dp_m\,dp. \tag{12}$$

Formula (12) makes it possible to transform the constitutive relation of problem (1) in the following manner:

$$\sigma_{ij}(x,t)=\sum_{n=1}^{\infty}\underbrace{\int_0^t\cdots\int_0^t}_{n}\Big\{\underbrace{\int\limits_{\Omega_p}\cdots\int\limits_{\Omega_{pn}}}_{n+1} R^{(*)}_{(ij)}(x,p,p_1,\ldots,p_n)\varphi_{l_1}(p,t)\prod_{m=1}^{n}\varphi_l^{+}(p_m,\tau_m)\,dp_m\,dp\Big\}\prod_{\xi=1}^{n}\varepsilon_{i_\xi j_\xi}(\tau_\xi)\,d\tau_\xi.$$

Considering (12) and the fact that the integral transforms with the kernels $\varphi_l(p,t)$ and $\varphi_l^{+}(p,t)$ are one-to-one transforms, we obtain

$$\sigma_{ij}(x,t)=\sum_{n=1}^{\infty}\int\limits_{\Omega_p}\Big\{\underbrace{\int\limits_{\Omega_{p_1}}\cdots\int\limits_{\Omega_{pn}}}_{n} R^{(*)}_{(ij)}(x,p,p_1,\ldots,p_n)\prod_{\xi=1}^{n}\varepsilon^{*}_{i_\xi j_\xi}(x,p_\xi)\,dp_\xi\Big\}\varphi_l(p,t)\,dp. \tag{13}$$

Let $\sigma_{ij}(x,t)$ also be the integral image of the inverse image $\sigma^{*}_{ij}(x,p)$:

$$\sigma_{ij}(x,t)=\int\limits_{\Omega_p}\sigma^{*}_{ij}(x,p)\varphi_l(p,t)\,dp. \tag{14}$$

It then follows from (13) and (14) that

$$\sigma^{*}_{ij}(x,p)=\sum_{n=1}^{\infty}\underbrace{\int\limits_{\Omega_{p_1}}\cdots\int\limits_{\Omega_{pn}}}_{n} R^{(*)}_{(ij)}(x,p,p_1,\ldots,p_n)\prod_{\xi=1}^{n}\varepsilon^{*}_{i_\xi j_\xi}(x,p_\xi)\,dp_\xi.$$

Assuming that $X_j(x,t)$, $P_i(x,t)$, and $u_i^0(x,t)$ are also the integral images of the quantities $X_j^{*}(x,p)$, $P_i^{*}(x,p)$, and $u_i^{*0}(x,p)$:

$$X_j(x,t)=\int\limits_{\Omega_p}X_j^{*}(x,p)\varphi_l(p,t)\,dp,\quad P_i(x,t)=\int\limits_{\Omega_p}P_i^{*}(x,p)\varphi_l(p,t)\,dp,$$

$$u_i^0(x,t)=\int\limits_{\Omega_p}u_i^{*0}(x,p)\varphi_l(p,t)\,dp,$$

we obtain

$$\frac{\partial\sigma^{*}_{ij}(x,p)}{\partial x_j}+X_i^{*}(x,p)=0,\quad \sigma^{*}_{ij}(x,p)n_j(x)=P_i^{*}(x,p),\quad x\in S_\sigma,\quad u_i^{*}(x,p)=u_i^{*0}(x,p),\quad x\in S_u.$$

Thus, the boundary-value problem of viscoelasticity (1) becomes a boundary-value problem for images:

$$\frac{\partial\sigma^{*}_{ij}(x,p)}{\partial x_j}+X_i^{*}(x,p)=0,\quad \varepsilon^{*}_{ij}(x,p)=\frac{1}{2}\Big\{\frac{\partial u_i^{*}(x,p)}{\partial x_j}+\frac{\partial u_j^{*}(x,p)}{\partial x_i}\Big\},$$

$$\sigma^{*}_{ij}(x,p)=\sum_{n=1}^{\infty}\underbrace{\int\limits_{\Omega_{p_1}}\cdots\int\limits_{\Omega_{pn}}}_{n} R^{(*)}_{(ij)}(x,p,p_1,\ldots,p_n)\prod_{\xi=1}^{n}\varepsilon^{*}_{i_\xi j_\xi}(x,p_\xi)\,dp_\xi, \tag{15}$$

$$\sigma^{*}_{ij}(x,p)n_j(x)=P_i^{*}(x,p),\quad x\in S_\sigma,\quad u_i^{*}(x,p)=u_i^{*0}(x,p),\quad x\in S_u.$$

Obviously, the boundary-value problem (15) does not have a time parameter, i.e., it is elastic for the images. Thus, by using integral transforms, we have reduced the given class of problems of the nonlinear nonuniform anisotropic theory of viscoelasticity to the corresponding class of problems of the theory of elasticity.

The choice of the pair $\varphi_l(p,t)$ and $\varphi_l^+(p,t)$ of direct and inverse integral transforms in each particular case is determined by the conditions that must be satisfied by the sought solutions of the initial viscoelastic problem $\sigma_{ij}(x,t)$, $u_i(x,t)$, and $\varepsilon_{ij}(x,t)$ and by the class of viscoelastic materials being described, i.e., the relaxation kernels $R_{(ij)}(x,t,\tau_1,\dots,\tau_n)$. For example, if a viscoelastic body occupies a finite volume, $\sigma_{ij}(x,t)$, $u_i(x,t)$, $\varepsilon_{ij}(x,t)$ and $R_{(ij)}(x,t,\tau_1,\dots,\tau_n)$ are assumed to be continuous with respect to all of their arguments and the problem is examined on a finite time interval, then Fourier transforms can be used as the direct and inverse integral transforms. When the behavior of an viscoelastic material is examined on a semi-infinite straight line [$t\in[0,+\infty)$], it becomes necessary to use direct and inverse Laplace transforms.

In making a transition from the viscoelastic boundary-value problem (1) to the corresponding elastic problem (15), the constitutive equation (15) can be simplified considerably in some situations. For example, this might occur in the case where $R^{(*)}_{(ij)}(x,p,p_1,\dots,p_n)$ satisfies the relation

$$R^{(*)}_{(ij)}(x,p,p_1,\dots,p_n)=R^{(*)}_{(ij)}(x,p)\prod_{\nu=1}^{n}G(p,p_\nu). \tag{16}$$

Here $R^{(*)}_{(ij)}(x,p)$ are the components of certain tensors and functions of the parameters $x$ and $p$, and $G(p,p_\nu)$ are the kernels of the unitary integral transformations. In this case, integral transforms with the kernels $\varphi_l(p,t)$ and $\varphi_l^+(p,t)$ will be referred to as optimum transforms. Here the following theorem proves to be valid.

**Theorem 2.** *Optimum transforms reduce the viscoelastic boundary-value problem* (1) *to a boundary-value problem of nonlinear elasticity for the inverse images*

$$\frac{\partial\sigma^*_{ij}(x,p)}{\partial x_j}+X^*_i(x,p)=0,\quad \varepsilon^*_{ij}(x,p)=\frac{1}{2}\left\{\frac{\partial u^*_i(x,p)}{\partial x_j}+\frac{\partial u^*_j(x,p)}{\partial x_i}\right\},$$

$$\sigma^*_{ij}(x,p)=\sum_{n=1}^{\infty}R^{(*)}_{(ij)}(x,p)\prod_{\xi=1}^{n}\varepsilon^*_{i_\xi j_\xi}(x,p), \tag{17}$$

$$\sigma^*_{ij}(x,p)n_j(x)=P^*_i(x,p),\quad x\in S_\sigma,\quad u^*_i(x,p)=u^{*0}_i(x,p),\quad x\in S_u.$$

**Proof.** To prove the theorem, it is sufficient to establish the validity of the constitutive equation of the boundary-value problem (17). In fact, substituting (16) into (15), we obtain

$$\sigma^*_{ij}(x,p)=\sum_{n=1}^{\infty}\underbrace{\int_{\Omega_{p_1}}\cdots\int_{\Omega_{p_n}}}_{n}R^{(*)}_{(ij)}(x,p)\prod_{\nu=1}^{n}G(p,p_\nu)\prod_{\xi=1}^{n}\varepsilon^*_{i_\xi j_\xi}(x,p_\xi)\,dp_\xi. \tag{18}$$

Integrating in (18), we have

$$\sigma^*_{ij}(x,p)=\sum_{n=1}^{\infty}R^{(*)}_{(ij)}(x,p)\prod_{\xi=1}^{n}\varepsilon^*_{i_\xi j_\xi}(x,p).$$

We shall determine the class of viscoelastic materials (relaxation kernels) for which relation (16) holds true, i.e., we shall determine the possibility of reducing the initial problem to the nonlinear elastic problem (17).

**Theorem 3.** *In order to be able to reduce the boundary-value problem* (1) *to the boundary-value problem* (17) *by means of integral transforms with the kernels $\varphi_l(p,t)$ and $\varphi_l^+(p,t)$, it is necessary and sufficient that the relaxation kernels be represented in the form*

$$R_{(ij)}(x,t,\tau_1,\dots,\tau_n)=\int_{\Omega_p}R^{(*)}_{(ij)}(x,p)\varphi_{l_1}(p,t)\prod_{\nu=1}^{n}\varphi_l^+(p,\tau_\nu)\,dp. \tag{19}$$

Necessity. Let the boundary-value problem (1) be reduced to problem (17) by integral transforms. Since the operations of differentiation with respect to the coordinates and integration over time are transposed in the given case, it suffices to prove this just for the constitutive equations

$$\sigma^*_{ij}(x,p) = \sum_{n=1}^{\infty} R^{(*)}_{(ij)}(x,p) \prod_{\xi=1}^{n} \varepsilon^*_{i_\xi j_\xi}(x,p). \tag{20}$$

Considering that $\varepsilon^*_{i_\xi j_\xi}(x,p) = \int\limits_{\Omega_{\tau_\xi}} \varepsilon_{i_\xi j_\xi}(x,\tau_\xi)\varphi_l^+(p,\tau_\xi)\,d\tau_\xi$, we find from (20) that

$$\sigma^*_{ij}(x,p) = \sum_{n=1}^{\infty} \underbrace{\int\limits_{\Omega_{\tau_\xi}} \dots \int\limits_{\Omega_{\tau_\xi}}}_{n} \left\{ R^{(*)}_{(ij)}(x,p) \prod_{\nu=1}^{n} \varphi_l^+(p,\tau_\nu) \right\} \prod_{\xi=1}^{n} \varepsilon_{i_\xi j_\xi}(x,\tau_\xi)\,d\tau_\xi. \tag{21}$$

Multiplying (21) by $\varphi_{l_1}(p,t)$ and integrating over $p$, we obtain

$$\sigma_{ij}(x,t) = \int\limits_{\Omega_p} \sigma^*_{ij}(x,p)\varphi_{l_1}(p,t)\,dp = \sum_{n=1}^{\infty} \underbrace{\int\limits_{\Omega_{\tau_\xi}} \dots \int\limits_{\Omega_{\tau_\xi}}}_{n} \left[ \int\limits_{\Omega_p} R^{(*)}_{(ij)}(x,p)\varphi_{l_1}(p,t) \prod_{\nu=1}^{n} \varphi_l^+(p,\tau_\nu)\,dp \right] \prod_{\xi=1}^{n} \varepsilon_{i_\xi j_\xi}(x,\tau_\xi)\,d\tau_\xi.$$

The expression in the square brackets is the kernel of the integral operator of the constitutive relations

$$R_{(ij)}(x,t,\tau_1,\dots,\tau_n) = \int\limits_{\Omega_p} R^{(*)}_{(ij)}(x,p)\varphi_l(p,t) \prod_{\xi=1}^{n} \varphi_l^+(p,\tau_\xi)\,dp.$$

Sufficiency. As before, it turns out to be sufficient to prove that the constitutive relation of problem (1) converges to the constitutive relation of problem (17). In fact, substituting (19) into (1), we obtain

$$\sigma_{ij}(x,t) = \sum_{n=1}^{\infty} \underbrace{\int\limits_{\Omega_{\tau_\xi}} \dots \int\limits_{\Omega_{\tau_\xi}}}_{n} \left[ R^{(*)}_{(ij)}(x,p)\varphi_{l_1}(p,t) \prod_{\nu=1}^{n} \varphi_l^+(p,\tau_\nu)\,dp \right] \prod_{\xi=1}^{n} \varepsilon_{i_\xi j_\xi}(x,\tau_\xi)\,d\tau_\xi.$$

Integrating over all $\tau_\xi$ and considering that $\int\limits_{\Omega_{\tau_\xi}} \varepsilon_{i_\xi j_\xi}(x,\tau_\xi)\varphi_l^+(p,\tau_\xi)\,d\tau_\xi = \varepsilon^*_{i_\xi j_\xi}(x,p)$, we arrive at the equality

$$\sigma^*_{ij}(x,p) = \sum_{n=1}^{\infty} R^{(*)}_{(ij)}(x,p) \prod_{\nu=1}^{n} \varepsilon^*_{i_\nu j_\nu}(x,p).$$

There are other transforms [2–4] that reduce general relations of type (1) for piecewise-degenerate kernels to a problem of nonlinear elasticity.

Let us illustrate the proposed method by solving a specific problem, namely, the problem of the stress-strain state of an infinite strip that is composed of an aging material and is compressed by two distributed forces. The thickness of the strip is $2b$, and the forces on its surface are distributed in accordance with known laws:

$$\frac{\partial \sigma_{ij}(x,t)}{\partial x_i} = 0, \quad \varepsilon_{ij}(x,t) = \frac{1}{2}\left\{ \frac{\partial u_i(x,t)}{\partial x_j} + \frac{\partial u_j(x,t)}{\partial x_i} \right\},$$

$$\begin{aligned} \varepsilon_{ij}(x,t) &= (1+\nu)\left[ (I-L)\frac{\sigma_{ij}(x,t)}{E_1} \right] - \nu\delta_{ij}(I-L)\left( \frac{\sigma_{mm}(x,t)}{E_1} \right) \\ &\quad + \beta(I-L)\left[ \frac{\sigma_{\alpha\beta}\sigma_{\alpha\beta}\delta_{ij}}{E_2} + \frac{2\sigma_{\alpha\alpha}(x,t)\sigma_{ij}(x,t)}{E_2} \right], \end{aligned} \tag{22}$$

$$\sigma_{ij}(x,t)n_j(x) = P_i(x,t) = \begin{cases} f_1(y), & x \in S_1, \\ f_2(y), & x \in S_2. \end{cases}$$

Here $\nu$ is the Poisson ratio, $E_1$ is Young's modulus, $E_2$ is the modulus of quadratically nonlinear elasticity, and $\beta$ is a material constant. The operator $(I - L)$ is expressed by the formula

$$(I - L)\frac{\sigma_{ij}}{E_1} = \frac{\sigma_{ij}(x,t)}{E_1(t)} - \int_0^t \frac{\sigma_{ij}(x,\tau)}{E_2(\tau)} K'(t, t-\tau)\, d\tau. \tag{23}$$

The operator $(I - L)$ has an inverse [5]. We write (23) in the form

$$(I - L)\frac{\sigma_{ij}}{E_1} = \int_0^\infty \left\{ G(t,\tau)\frac{\sigma_{ij}(x,\tau)}{E_1(\tau)} - \frac{\sigma_{ij}(x,\tau)}{E_1(\tau)} K'(t, t-\tau) \right\} d\tau. \tag{24}$$

Here $G(t,\tau)$ is the kernel of a unitary integral transformation, and the kernel $K'(t, t-\tau)$ is assumed to be equal to zero for moments $\tau$ greater than $t$. We rewrite (24) in the form

$$(I - L)\frac{\sigma_{ij}}{E_1} = \int_0^\infty \frac{\sigma_{ij}(x,\tau)}{E_1(\tau)} \Big[ G(t,\tau) - K'(t, t-\tau) \Big] d\tau.$$

We denote $G(t,\tau) - K'(t, t-\tau) = \varphi_l(t,\tau)$.

Thus, in accordance with (5), the inverse transform is known for the kernel $\varphi_l(t,\tau)$. We denote this transform by $\varphi_l^+(t,\tau)$. As a result, we have the following as the constitutive equation for the boundary-value problem (22):

$$\varepsilon_{ij}(x,t) = \int_0^\infty \left[ \frac{(1+\nu)\sigma_{ij}}{E_1} - \nu\delta_{ij}\frac{\sigma_{mm}}{E_1} + \beta\frac{\sigma_{\alpha\beta}\sigma_{\alpha\beta}\delta_{ij}}{E_2} + \frac{2\sigma_{\alpha\alpha}\sigma_{ij}}{E_2} \right] \{G(t,\tau) - K'(t, t-\tau)\}\, d\tau.$$

Using Eqs. (6) and (9)–(11), for problem (22) we obtain

$$\begin{gathered} \frac{\partial\sigma_{ij}(x,t)}{\partial x_j} = 0, \qquad \varepsilon_{ij}^*(x,t) = \frac{1}{2}\left\{ \frac{\partial u_i^*(x,t)}{\partial x_j} + \frac{\partial u_j^*(x,t)}{\partial x_i} \right\}, \\ \varepsilon_{ij}^*(x,t) = \frac{1+\nu}{E_1(\tau)}\sigma_{ij}(x,t) - \nu\delta_{ij}\frac{\sigma_{mm}}{E_1} + \beta\frac{\sigma_{\alpha\beta}\sigma_{\alpha\beta}\delta_{ij}}{E_2} + \frac{2\sigma_{\alpha\alpha}\sigma_{ij}}{E_2}, \\ \sigma_{ij}(x,t)n_j(x) = P_i(x,t) = \begin{cases} f_1(y), & x \in S_1, \\ f_2(y), & x \in S_2. \end{cases} \end{gathered} \tag{25}$$

The boundary-value problem (25) is a nonlinear elastic problem. It is of interest that only the strains, rather than the stresses, were transformed in this case. Thus, the stresses of the elastic problem (25) give the stresses of the viscoelastic problem. The constitutive relations are used to determine the strain tensor, which is then used to determine the displacement vector.

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